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Complete f-Moment Convergence for Sung's Type Weighted Sums of Negatively Superadditive Dependent Random Variables*

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Abstract: In this paper, by utilizing the Marcinkiewicz-Zygmund inequality and Rosenthal-type inequality of negatively superadditive dependent (NSD) random arrays and truncated method, we investigate the complete f-moment convergence of NSD random variables. We establish and improve a general result on the complete f-moment convergence for Sung's type randomly weighted sums of NSD random variables under some general assumptions. As an application, we show the complete consistency for the randomly weighted estimator in a nonparametric regression model based on NSD errors.

Keywords: Marcinkiewicz-Zygmund inequality; Rosenthal-type inequality; Sung's type randomly weighted sums; negatively superadditive dependent random variables; complete f-moment convergence

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1 Inroduction

The concept of complete convergence was introduced by Hsu and Robbins[1] as follows: a sequence $\{X_n, n \geqslant 1\}$ of random variables is said to converge completely to the constant θ if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathsf{P}\left(|X_n - \theta| > \varepsilon\right) < \infty.$$

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By the Borel-Cantelli lemma, this implies that $X_n \to \theta$ almost surely, and so complete convergence is a stronger concept than almost sure convergence. Hsu and Robbins[1] proved that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summands is finite.

Recently, Sung[2] obtained the following complete convergence for weighted sums of ρ^*-mixing random variables.

Theorem 1 Let $1/2 < \alpha \leqslant 1, p > 1/\alpha, \{X_n, n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $\mathsf{E}(X_1) = 0$. Assume that $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ is an array of positive real numbers satisfying

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \text{ for some } q > p. \tag{1}$$

If $\mathsf{E}(|X_1|^p) < \infty$, then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathsf{P}\left(\max_{1 \leqslant k \leqslant n} \left|\sum_{i=1}^{k} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{2}$$

Conversely, if (2) holds for any array $\{a_{ni}\}$ satisfying (1), then $\mathsf{E}(|X_1|^p) < \infty$.

Chow[3] defined the concept of complete moment convergence as follows:

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables, and $\{a_n, n \geqslant 1\}$, $\{b_n, n \geqslant 1\}$ be two sequences of positive real numbers. If for any $\varepsilon > 0$ and $q > 0$, we have

$$\sum_{n=1}^{\infty} a_n \mathsf{E}\left(\left\{b_n^{-1}|X_n| - \varepsilon\right\}_+^q\right) < \infty,$$

where $a_+ = \max\{0, a\}, a_+^q = (a_+)^q$, then $\{X_n, n \geqslant 1\}$ is said to be complete q-th moment convergent.

Wu et al.[4] extended the complete convergence of Sung's type weighted sums of sequences of ρ^*-mixing random variables to complete moment convergence. Recently, Wu et al.[5] introduced the concept of complete f-moment convergence, which is stronger than complete moment convergence and complete convergence.

Let $\{S_n, n \geqslant 1\}$ be a sequence of random variables, $\{a_n, n \geqslant 1\}$ be a sequence of positive constants and $f: \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function with $f(0) = 0$. Then we say that $\{S_n, n \geqslant 1\}$ converges f-moment completely if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n \mathsf{E}\left(f\left(\{|S_n| - \varepsilon\}_+\right)\right) < \infty.$$

It is straightforward to verify that complete f-moment convergence reduces to complete moment convergence if $f(u) = u^q$, and to complete convergence if $a_n = 1, n \geqslant 1$, and $f(u) = u$.

Afterwards, many authors were devoted to studying the complete f-moment convergence and obtained numerous meaning results. For example, Lu et al.[6] studied complete f-moment convergence for widely orthant dependent (WOD) random variables and gave its application in nonparametric models. Lu et al.[7] obtained the complete f-moment convergence for sums of arrays of rowwise END random variables under sub-linear expectation space. Wang et al.[8] explored complete f-moment convergence for NSD random variables. Especially, Wang and Wang[9] obtained a result on complete f-moment convergence for Sung's type weighted sums of extended negatively dependent (END) random variables under some general conditions as follows.

Theorem 2 Let $v>0, \alpha>1/2, \alpha p>0, q>p \vee v \geqslant 1, \alpha(p \vee v) \geqslant 1$, and $\{X_n, n \geqslant 1\}$ be a sequence of END random variables with mean zero, which is stochastically dominated (refering to definition 4) by a random variable X. Let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n).$$

If

$$\begin{cases} \mathsf{E}\left(|X|^p\right)<\infty, & v<p, \\ \mathsf{E}\left(|X|^p\right)\log(1+|X|)<\infty, & v=p, \\ \mathsf{E}\left(|X|^v\right)<\infty, & v>p, \end{cases}$$

then for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathsf{E}\left(f\left(\left\{\max_{1 \leqslant k \leqslant n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| / n^{\alpha}\right\}-\varepsilon\right)_{+}\right)<\infty,$$

where $p \vee v=\max\{p, v\}$.

So far, the corresponding research results on the complete f-moment convergence for Sung's type randomly weighted sums of rowwise NSD random variables have not been obtained. Now let us recall the concepts of superadditive function and negatively superadditive dependence. The concept of superadditive function was introduced by Kemperman[10] as follows.

Definition 1 A function $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$ is called superadditive if $\phi(x \vee y)+\phi(x \wedge y) \geqslant \phi(x)+\phi(y)$ for all $x, y \in \mathbb{R}^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

The following concept of NSD random variables, which is weaker than negative association (NA) was introduced by Hu[11].

Definition 2 A random vector $X=(X_1,X_2,\cdots,X_n)$ is said to NSD if

$$\mathsf{E}\left(\phi\left(X_1,X_2,\cdots,X_n\right)\right)\leqslant\mathsf{E}\left(\phi\left(X_1^*,X_2^*,\cdots,X_n^*\right)\right),\tag{3}$$

where $X_1^*,X_2^*,\cdots,X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i, and ϕ is a superadditive function such that the expectations in (3) exist.

A sequence $\{X_n,n\geqslant 1\}$ of random variables is said to be NSD if for all $n\geqslant 1$, $(X_1,X_2,\cdots,X_n)$ is NSD. An array $\{X_{ni},i\geqslant 1,n\geqslant 1\}$ of random variables is said to be NSD if for all $n\geqslant 1,\{X_{ni},i\geqslant 1\}$ is NSD.

Shen et al.[12] obtained the almost sure convergence for NSD sequences and the strong stability for weighted sums of NSD random variables, which extend the corresponding results for independent sequences and NA sequences without necessarily adding any extra condition. Wang et al.[13] presented the Rosenthal-type maximal inequalities and Kolmogorov-type exponential inequality for NSD random variables, studied the complete convergence for arrays of rowwise NSD random variables and weighted sums of arrays of rowwise NSD random variables. Wang et al.[14] obtained the complete convergence for weighted sums of NSD random variables and its application in the the errors-in-variables (EV) regression model. Xue et al.[15] investigated the complete moment convergence for maximal partial sum of NSD random variables under some more general conditions. Shen et al.[16] investigated the complete convergence and complete moment convergence for arrays of rowwise NSD random variables and presented some sufficient conditions to prove the complete convergence and the complete moment convergence.

In addition, we need to present the concept of slowly varying function and stochastic domination as follows.

Definition 3 A real-valued function $l(x)$, positive and measurable on $(0,\infty)$, is said to be a slowly varying function at ∞ if

$$\lim_{x\to\infty}\frac{l(\lambda x)}{l(x)}=1 \text{ for each } \lambda>0.$$

Definition 4 A sequence $\{X_n,n\geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X, if for all $x\geqslant 0$, there exists a positive constant C such that

$$\sup_{n\geqslant 1}\mathsf{P}\left(|X_n|>x\right)\leqslant C\mathsf{P}(|X|>x).$$

Wang et al.[17] obtained complete moment convergence of Sung's type weighted sums of sequences of NSD random variables.

Theorem 3 Let $0<p<2,\alpha p\geqslant 1,\{X_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ be a triangular array of rowwise NSD random variables stochastically dominated by a random variable X, $\mathsf{E}(X_{ni}) = 0$. Let $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a triangular array of rowwise independent random variables, which is independent of $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$, and

$$\sum_{i=1}^{n} \mathsf{E}(|A_{ni}|^q) = O(n), \quad 1 \vee p < q \leqslant 2.$$

If

$$\begin{cases} \mathsf{E}(|X|^p)\, l\left(|X|^{1/\alpha}\right) < \infty, & 1 < p < 2, \\ \mathsf{E}(|X|)\, l\left(|X|^{1/\alpha}\right) \log(1+|X|) < \infty, & p = 1, \\ \mathsf{E}(|X|) < \infty, & 0 < p < 1, \end{cases}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p - 2 - \alpha} l(n) \mathsf{E}\left[\left(\max_{1 \leqslant k \leqslant n} \left|\sum_{i=1}^{k} A_{ni} X_{ni}\right| - \varepsilon n^{\alpha}\right)_+\right] < \infty, \tag{4}$$

and thus

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) \mathsf{P}\left(\max_{1 \leqslant k \leqslant n} \left|\sum_{i=1}^{k} A_{ni} X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty. \tag{5}$$

Theorem 4 Let $p \geqslant 2, \alpha p \geqslant 1, \{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a triangular array of rowwise NSD random variables stochastically dominated by a random variable X, $\mathsf{E}(X_{ni}) = 0, \mathsf{E}(|X|^p)\, l\left(|X|^{1/\alpha}\right) < \infty$. Let $\{A_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a triangular array of rowwise independent random variables, which is independent of $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$, and

$$\sum_{i=1}^{n} \mathsf{E}(|A_{ni}|^q) = O(n), \quad q > 2(\alpha p - 1)/(2\alpha - 1).$$

Then for any $\varepsilon > 0$, (4) and (5) also hold.

Inspired by the literature above, especially [9] and [17], we aim to establish some sufficient conditions for complete f-moment convergence for Sung's type randomly weighted sums of NSD random variables which extend and improve the results of complete moment convergence in Theorems 3 and 4.

This paper is organized as follows. Some preliminary lemmas and inequalities for NSD random variables are provided in Section 2. The main result and some corollaries are given in Section 3. An application in a nonparametric regression model based on NSD errors is presented in Section 4. Throughout the paper, we will denote by C a positive generic constant, which may be different in various places.

2 Preliminary lemmas

In this section, we give some important lemmas which will be used to prove our main results. The following one was presented by Hu[11].

Lemma 5 Let $(X_1, X_2, \cdots, X_n)$ be NSD.

(i) $(-X_1, -X_2, \cdots, -X_n)$ is also NSD.

(ii) If $g_1, g_2, \cdots, g_n$ are all nondecreasing functions, then $(g_1(X_1), g_2(X_2), \cdots, g_n(X_n))$ is NSD.

The next one comes from Wang et al.[17].

Lemma 6 If $\{X_i, 1 \leqslant i \leqslant n\}$ is a sequence of NSD random variables, $\{Y_i, 1 \leqslant i \leqslant n\}$ is a sequence of non-negative independent random variables, and the two sequences are independent with each other. Then $\{X_iY_i, 1 \leqslant i \leqslant n\}$ is still a sequence of NSD random variables.

The first inequality in Lemma 7 is the Marcinkiewicz-Zygmund inequality and the second one is the Rosenthal-type inequality for NSD random variables, which can be both found in [13].

Lemma 7 Let $\{X_n, n \geqslant 1\}$ be a sequence of NSD random variables with $\mathsf{E}(X_n) = 0$, $\mathsf{E}(|X_n|^s) < \infty$ for each $i \geqslant 1$, and then there exists a positive constant C depending only s such that

$$\mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^s\right) \leqslant C\sum_{i=1}^{n}\mathsf{E}(|X_i|^s), \quad 1 < s \leqslant 2,$$

$$\mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} X_i\right|^s\right) \leqslant C\left[\sum_{i=1}^{n}\mathsf{E}(|X_i|^s) + \left(\sum_{i=1}^{n}\mathsf{E}\left(|X_i|^2\right)\right)^{s/2}\right], \quad s > 2.$$

The following lemma comes from Wu et al.[4].

Lemma 8 Let $\{Y_i, 1 \leqslant i \leqslant n\}, \{Z_i, 1 \leqslant i \leqslant n\}$ be two sequences of random variables. Then for any $\varepsilon > 0, a > 0, q > r > 0$, the following inequality holds

$$\mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}(Y_i + Z_i)\right| - \varepsilon a\right)^r\right] \leqslant C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right)a^{r-q}\mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} Y_i\right|^q\right) + C_r\mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} Z_i\right|^r\right),$$

where $C_r = 1$ if $0 < r \leqslant 1$ or $C_r = 2^{r-1}$ if $r > 1$.

The following lemma is an important property of stochastic domination. The first inequality and the second inequality can be found in [18] and [19].

Lemma 9 Let $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $m > 0$, and $d > 0$, the following two statements hold:

$$\mathsf{E}\left(|X_{ni}|^m\right) I\left(|X_{ni}| \leqslant d\right) \leqslant C_1\left[\mathsf{E}\left(|X|^m\right) I(|X| \leqslant d) + d^m \mathsf{P}(|X| > d)\right],$$

$$\mathsf{E}\left(|X_{ni}|^m\right) I\left(|X_{ni}| > d\right) \leqslant C_2 \mathsf{E}\left(|X|^m\right) I(|X| > d),$$

where C_1 and C_2 are positive constants. Thus, $\mathsf{E}\left(|X_{ni}|^m\right) \leqslant C\mathsf{E}\left(|X|^m\right)$.

The last lemma was given by Bai and Su[20].

Lemma 10 Let X be a random variable and $l(\cdot) > 0$ be a slowly varying function, the following three conclusions hold:

(i) if $\alpha \geqslant 0, \gamma > 0$, then

$$\sum_{n=1}^{\infty} n^{-1} \mathsf{E}\left(|X|^\alpha\right) I\left(|X| > n^\gamma\right) \leqslant C\mathsf{E}\left(|X|^\alpha\right) \log(1 + |X|);$$

(ii) if $\beta > -1, \alpha \geqslant 0, \gamma > 0$, then

$$\sum_{n=1}^{\infty} n^\beta l(n) \mathsf{E}\left(|X|^\alpha\right) I\left(|X| > n^\gamma\right) \leqslant C\mathsf{E}\left(|X|^{\alpha+(\beta+1)/\gamma}\right) l\left(\left|X^{1/\gamma}\right|\right);$$

(iii) if $\beta < -1, \alpha \geqslant 0, \gamma > 0$, then

$$\sum_{n=1}^{\infty} n^\beta l(n) \mathsf{E}\left(|X|^\alpha\right) I\left(|X| \leqslant n^\gamma\right) \leqslant C\mathsf{E}\left(|X|^{\alpha+(\beta+1)/\gamma}\right) l\left(\left|X^{1/\gamma}\right|\right).$$

3 The main result and its proof

Firstly, we introduce a function

$$f : \mathbb{R}^+ \to \mathbb{R}^+,$$

which is increasing and continuous with $f(0) = 0$.

Let

$$g : \mathbb{R}^+ \to \mathbb{R}^+,$$

$g(t)$ be the inverse function of $f(t)$, and satisfy $g(f(t)) = t, t \geqslant 0$. Assume that for some positive constants $\delta, r > 1$, the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$\int_{f(\delta)}^{\infty} g^{-r}(t) \mathrm{d}t < \infty. \tag{6}$$

Remark 1 Concerning the conditions on f and g, we can refer the readers to [5–9], for instance. Note that (6) is easily satified, and the most interesting case is to consider the function $f(t)=t^q, t\geqslant 0, 0<q<r$.

Using the above functions f and g, we give the following main result and its proof.

Theorem 11 Let $\alpha>1/2, \alpha p\geqslant 1, 1<r\leqslant 2, p>r, \{X_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be a triangular array of NSD random variables with mean zero, which is stochastically dominated by a random variable X. Let $\{A_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be a triangular array of rowwise independent random variables, which is independent of $\{X_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$, and

$$\sum_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^q\right)=O(n), \quad q>(p\vee 2(p\alpha-1)/(2\alpha-1)). \tag{7}$$

If

$$\mathsf{E}\left(|X|^p\right)l\left(|X|^{1/\alpha}\right)<\infty, \tag{8}$$

then for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{E}\left(f\left(\{|S_n-\varepsilon|\}_+\right)\right)<\infty,$$

where $S_n=\max\limits_{1\leqslant k\leqslant n}\left|\sum\limits_{i=1}^{k}A_{ni}X_{ni}\right|/n^{\alpha}, 1\leqslant i\leqslant n, n\geqslant 1$.

Proof For any $\varepsilon>0$, it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{E}\left(f\left(\{|S_n|-\varepsilon\}_+\right)\right)\\
=&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_0^{\infty}\mathsf{P}\left[f\left(\{|S_n|-\varepsilon\}_+\right)>t\right]\mathrm{d}t\\
=&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_0^{\infty}\mathsf{P}\left[\{|S_n|-\varepsilon\}_+>g(t)\right]\mathrm{d}t\\
=&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_0^{f(\delta)}\mathsf{P}\left(|S_n|>\varepsilon+g(t)\right)\mathrm{d}t\\
&+\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_{f(\delta)}^{\infty}\mathsf{P}\left(|S_n|>\varepsilon+g(t)\right)\mathrm{d}t\\
\triangleq&H_1+H_2.
\end{aligned}$$

According to Theorems 3 and 4, we can easily see that

$$H_1\leqslant\int_0^{f(\delta)}\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{P}\left(|S_n|>\varepsilon\right)\mathrm{d}t\leqslant f(\delta)\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{P}\left(|S_n|>\varepsilon\right)<\infty.$$

Thus, in order to prove Theorem 11, we need to show $\mathrm{H}_2<\infty$. By Markov's inequality and (6), we have

$$\begin{aligned}H_2 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} A_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_{+}^{r}\right] \int_{f(\delta)}^{\infty} g^{-r}(t)\mathrm{d}t \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} A_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_{+}^{r}\right].\end{aligned}$$

Let us make some notations for $1\leqslant i\leqslant n, n\geqslant 1$ as follows.

$$Y_{ni} = A_{ni}\left(-n^{\alpha} I\left(X_{ni} < -n^{\alpha}\right) + X_{ni} I\left(\left|X_{ni}\right| \leqslant n^{\alpha}\right) + n^{\alpha} I\left(X_{ni} > n^{\alpha}\right)\right),$$

$$Z_{ni} = A_{ni}X_{ni} - Y_{ni} = A_{ni}\left(X_{ni} + n^{\alpha}\right) I\left(X_{ni} < -n^{\alpha}\right) + A_{ni}\left(X_{ni} - n^{\alpha}\right) I\left(X_{ni} > n^{\alpha}\right).$$

By $\mathsf{E}\left(X_{ni}\right) = 0$ and Lemma 8, we can get that

$$\begin{aligned}H_2 &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} A_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right)_{+}^{r}\right] \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} \left(Y_{ni} - \mathsf{E}\left(Y_{ni}\right) + Z_{ni} - \mathsf{E}\left(Z_{ni}\right)\right)\right| - \varepsilon n^{\alpha}\right)_{+}^{r}\right] \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left[\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} \left(Y_{ni} - \mathsf{E}\left(Y_{ni}\right) + Z_{ni} - \mathsf{E}\left(Z_{ni}\right)\right)\right| - \frac{\varepsilon}{2} n^{\alpha}\right)_{+}^{r}\right] \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} \left(Y_{ni} - \mathsf{E}\left(Y_{ni}\right)\right)\right|^{q}\right) \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \mathsf{E}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} \left(Z_{ni} - \mathsf{E}\left(Z_{ni}\right)\right)\right|^{r}\right) \\ &\triangleq H_3 + H_4.\end{aligned}$$

Next, according to Lemma 9, Lemma 10, (7) and (8), for $r<p<q$, it is easy to obtain

$$\begin{aligned}H_4 &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \sum_{i=1}^{n} \mathsf{E}\left(\left|Z_{ni} - \mathsf{E}\left(Z_{ni}\right)\right|^{r}\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \sum_{i=1}^{n} \mathsf{E}\left(\left|Z_{ni}\right|^{r}\right) \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha r} l(n) \sum_{i=1}^{n} \mathsf{E}\left(\left|A_{ni}\right|^{r}\right) \mathsf{E}\left(\left(\left|X_{ni}\right| - n^{\alpha}\right)^{r}\right) I\left(\left|X_{ni}\right| > n^{\alpha}\right) \\ &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha r} l(n) \mathsf{E}\left(|X|^{r}\right) I\left(|X| > n^{\alpha}\right)\end{aligned}$$

$$\leqslant C\mathsf{E}\left(|X|^p\right)l\left(|X|^{1/\alpha}\right)<\infty.$$

In order to prove $H_3<\infty$, we will divide it into two cases.

Case 1: $\alpha>1/2, \alpha p\geqslant 1, 1<p<2, p<q\leqslant 2$. It follows from Lemma 5 and 6 that $\{Y_{ni}-\mathsf{E}\left(Y_{ni}\right), 1\leqslant i\leqslant n, n\geqslant 1\}$ is still an array of rowwise NSD random variables. By Lemma 7, Lemma 9, C_r- inequality and Hölder's inequality, we can get

$$\begin{aligned}
H_3 &\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\sum_{i=1}^{n}\mathsf{E}\left(|Y_{ni}|^q\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\sum_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^q\right)\left[\mathsf{E}\left(|X|^q\right)I\left(|X|\leqslant n^{\alpha}\right)+n^{\alpha q}\mathsf{P}\left(|X|>n^{\alpha}\right)\right]\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha q}l(n)\mathsf{E}\left(|X|^q\right)I\left(|X|\leqslant n^{\alpha}\right)+C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)\mathsf{P}\left(|X|>n^{\alpha}\right)\\
&\triangleq H_{31}+H_{32}.
\end{aligned}$$

When $q>p$, by Lemma 10, we can obtain $H_{31}<\infty$. Next, we need to prove that $H_{32}<\infty$. Similar to the proof above of H_{31}, we have

$$\begin{aligned}
H_{32} &= C\sum_{n=1}^{\infty}n^{\alpha p-1}l(n)\sum_{m=n}^{\infty}\mathsf{P}\left(m^{\alpha}<|X|\leqslant(m+1)^{\alpha}\right)\\
&= C\sum_{m=1}^{\infty}\mathsf{P}\left(m^{\alpha}<|X|\leqslant(m+1)^{\alpha}\right)\sum_{n=1}^{m}n^{\alpha p-1}l(n)\\
&\leqslant C\sum_{m=1}^{\infty}m^{\alpha p}l(m)\mathsf{P}\left(m^{\alpha}<|X|\leqslant(m+1)^{\alpha}\right)\\
&\leqslant C\mathsf{E}\left(|X|^p\right)l\left(|X|^{1/\alpha}\right)<\infty.
\end{aligned}$$

Case $2: \alpha>1/2, p\geqslant 2, q>2(p\alpha-1)/(2\alpha-1)$.

By Lemma 7, when $q\geqslant 2(\alpha p-1)/(2\alpha-1)$, note that

$$\begin{aligned}
H_3 &\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\sum_{i=1}^{n}\mathsf{E}\left(|Y_{ni}|^q\right)+C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\left(\sum_{i=1}^{n}\mathsf{E}\left(Y_{ni}^2\right)\right)^{q/2}\\
&\triangleq H_{31}'+H_{32}'.
\end{aligned}$$

We obtain by Lemma 9, Lemma 10 and (7) that

$$\begin{aligned}
H_{31}' &\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\sum_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^q\right)\left(\mathsf{E}\left(|X|^q\right)I\left(|X|\leqslant n^{\alpha}\right)+n^{\alpha q}\mathsf{P}\left(|X|>n^{\alpha}\right)\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha q}l(n)\mathsf{E}\left(|X|^q\right)I\left(|X|\leqslant n^{\alpha}\right)+C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha}l(n)\mathsf{E}\left(|X|\right)I\left(|X|>n^{\alpha}\right)\\
&\leqslant 2C\mathsf{E}\left(|X|^p\right)l\left(|X|^{|1/\alpha}\right)<\infty.
\end{aligned}$$

Next, we go on to prove that $H'_{32}<\infty$. It follows from (7) that $\sum\limits_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^2\right)=O(n)$. When $p>2$, we have by (8) that $\mathsf{E}\left(|X|^2\right)<\infty$. By Hölder's inequality, we have

$$\begin{aligned}
H'_{32}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\left(\sum_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^2\right)\left(n^{2\alpha}\mathsf{P}\left(|X|>n^{\alpha}\right)+\mathsf{E}\left(|X|^2\right)I\left(|X|\leqslant n^{\alpha}\right)\right)\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\left(\sum_{i=1}^{n}\mathsf{E}\left(|A_{ni}|^2\right)\mathsf{E}\left(|X|^2\right)\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1+\alpha p-1-(\alpha-1/2)q}l(n)\left(\mathsf{E}\left(|X|^2\right)\right)^{q/2}\\
&<\infty.
\end{aligned}$$

When $p=2$, (8) holds. Thus, when $0<\delta<(2\alpha q-q-2\alpha p+2)/\alpha q$, $\mathsf{E}\left(|X|^{2-\delta}\right)<\infty$. By Lemma 9, we can get

$$\begin{aligned}
H'_{32}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\left(\sum_{i=1}^{n}|A_{ni}|^2\left(\mathsf{E}\left(|X|^2\right)I\left(|X|\leqslant n^{\alpha}\right)+n^{2\alpha}\mathsf{P}\left(|X|>n^{\alpha}\right)\right)\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)\left(\sum_{i=1}^{n}|A_{ni}|^2\left(\mathsf{E}\left(|X|^{2-\delta}\right)n^{\alpha\delta}I\left(|X|\leqslant n^{\alpha}\right)+n^{2\alpha}\mathsf{P}\left(|X|>n^{\alpha}\right)\right)\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)n^{q/2}\left(\mathsf{E}\left(|X|^{2-\delta}\right)n^{\alpha\delta}I\left(|X|\leqslant n^{\alpha}\right)+2n^{2\alpha}\mathsf{P}\left(|X|>n^{\alpha}\right)\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q}l(n)n^{q/2}\left(n^{\alpha\delta}+2n^{2\alpha}\frac{\mathsf{E}\left(|X|^{2-\delta}\right)}{n^{\alpha(2-\delta)}}\right)^{q/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha q+q/2+\alpha\delta q/2}l(n)<\infty.
\end{aligned}$$

This completes the proof of Theorem 11. □

Remark 2 In view of the proof above, the essential tools are the Rosenthal-type maximum inequality and the Marcinkiewicz-Zygmund type maximum inequality which are also available for independent sequence, NA sequence, END sequence, WOD sequence, α-mixing sequence, ϕ-mixing sequence, ϕ^*-mixing sequence, identically distributed ρ-mixing sequence, and so on. Hence, the main results also hold true for these sequences.

We introduce the same functions f and g as in Section 3 and satisfy (6). It is easy to obtain the following corollaries.

Corollary 12 Let $\alpha>1/2,\alpha p\geqslant 1,1<r<p,\{X_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ be a triangular array of NSD random variables with mean zero, which is stochastically dominated by a random variable X satisfying (8). Let $\{a_{ni},1\leqslant i\leqslant n,n\geqslant 1\}$ be an array of real numbers

satisfying

$$\sum_{i=1}^{n}|a_{ni}|^{q}=O(n)\quad q>(p\vee 2(p\alpha-1)/(2\alpha-1)). \tag{9}$$

Then for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{E}\left[f\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}-\varepsilon\right\}_{+}\right)\right]<\infty.$$

Thus,

$$\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|>\varepsilon n^{\alpha}\right)<\infty.$$

Proof By the condition of assumption, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{E}\left[f\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}-\varepsilon\right\}_{+}\right)\right]\\
&=\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_{0}^{\infty}\mathsf{P}\left(\left[f\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}-\varepsilon\right\}\right)\right]>t\right)\mathrm{d}t\\
&=\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_{0}^{\infty}\mathsf{P}\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}-\varepsilon\right\}_{+}>g(t)\right)\mathrm{d}t\\
&=\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_{0}^{\infty}\mathsf{P}\left[\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}>g(t)+\varepsilon\right]\mathrm{d}t\\
&\geqslant\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\int_{0}^{f(\varepsilon)}\mathsf{P}\left[\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|\Big/ n^{\alpha}>\varepsilon+\varepsilon\right]\mathrm{d}t\\
&\geqslant f(\varepsilon)\sum_{n=1}^{\infty}n^{\alpha p-2}l(n)\mathsf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_{ni}X_{ni}\right|>2\varepsilon n^{\alpha}\right).
\end{aligned}$$

This completes the proof of corollary 12. □

Remark 3 According to Corollary 12, it follows that the complete f-moment convergence is a stronger concept than the complete moment convergence.

Concerning the condition on $f(t)$, we point out that (6) holds trivially if we take $f(t)=t^{s}, t\geqslant 0, 1\leqslant s<r$. Hence, we can obtain complete moment convergence similar to Theorems 3 and 4 according to Theorem 11 and Corollary 12.

Corollary 13 Let $\alpha>1/2, \alpha p\geqslant 1, 1\leqslant s<2, s<p, \{X_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be a triangular array of NSD random variables with mean zero, which is stochastically dominated by a random variable X satisfying (8). Let $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ be an array

of real numbers satisfying (9). Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha s-2} l(n) \mathsf{E}\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| - \varepsilon n^{\alpha}\right\}_{+}^{s}\right) < \infty.$$

Thus,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathsf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right| > \varepsilon n^{\alpha}\right) < \infty.$$

Remark 4 Obviously, when $s = 1$, Corollary 13 is equivalent to Theorems 3 and 4, respectively. Thus, we extend the related results in [17].

Taking $a_{ni} \equiv a_i$ in Corollary 13, we have the following corollary.

Corollary 14 Let $\alpha > 1/2, \alpha p \geqslant 1, 1 \leqslant s < 2, s < p, \{X_i, i \geqslant 1\}$ be a sequence of NSD random variables with mean zero, which is stochastically dominated by a random variable X satisfying (8). Let $\{a_i, i \geqslant 1\}$ be an array of real numbers satisfying

$$\sum_{i=1}^{n} |a_i|^q = O(n), \quad q > (p \vee 2(p\alpha - 1)/(2\alpha - 1)).$$

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha v-2} l(n) \mathsf{E}\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{i}X_{i}\right| - \varepsilon n^{\alpha}\right\}_{+}^{s}\right) < \infty.$$

Thus,

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathsf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{i}X_{i}\right| > \varepsilon n^{\alpha}\right) < \infty.$$

If we take $1 \leqslant s \leqslant l < 2, \{X_i, i \geqslant 1\}, l(n) \equiv 1, p = 2l, \alpha = 1/l$ in Corollary 14, we can get the following result directly.

Corollary 15 Let $1 \leqslant s \leqslant l < 2, \{X_i, i \geqslant 1\}$ be a sequence of NSD random variables with mean zero, which is stochastically dominated by a random variable X satisfying (8). Let $\{a_i, i \geqslant 1\}$ be a sequence of real numbers satisfying

$$\sum_{i=1}^{n} |a_i|^q = O(n), \quad q > 2l/(2 - l).$$

If $\mathsf{E}\left(|X|^{2l}\right) < \infty$ holds, then for any $\varepsilon > 0$.

$$\sum_{n=1}^{\infty} n^{-v/l} \mathsf{E}\left(\left\{\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k} a_{i}X_{i}\right| - \varepsilon n^{1/l}\right\}_{+}^{s}\right) < \infty.$$

Therefore,

$$\sum_{n=1}^{\infty}\mathsf{P}\left(\max_{1\leqslant k\leqslant n}\left|\sum_{i=1}^{k}a_iX_i\right|>\varepsilon n^{1/l}\right)<\infty.$$

4 An application in nonparametric regression model

Consider the following nonparametric regression model:

$$Y_{ni}=h(x_{ni})+\epsilon_{ni}, i=1,2,\cdots,n, n\geqslant 1, \tag{10}$$

where the regression function $h(\cdot)$ is an unknown Borel measurable function defined on the compact set $A\subset R^m, m\geqslant 1$, x_{ni} are known design points from A, and ϵ_{ni} are random errors such that $(\epsilon_{n1},\epsilon_{n2},\cdots,\epsilon_{nn})$ have the same distribution as $(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$. We consider the following randomly weighted regression estimator of $h(\cdot)$:

$$h_n(x)=\sum_{i=1}^{n}\omega_{ni}(x)Y_{ni}, x\in A\subset R^m, \tag{11}$$

where $\omega_{ni}(x)=\omega_{ni}(x,x_{n1},x_{n2},\cdots,x_{nn}), i=1,2,\cdots,n$ are randomly weighted functions which are independent of $\epsilon_{ni}, i=1,2,\cdots,n$.

The above estimator with constant weight about the model (10) was first introduced by Stone[21] and next adapted by Georgiev[22] to the fixed design case. Up to now, the estimator has been studied by many authors; see, among others, [17,23–25].

In this subsection, let $c(h)$ denote all continuity points of the function h on A. The symbol $\|x\|$ denotes the Euclidean norm. For any point $x\in A$, we will consider the following assumptions on randomly weighted functions $\omega_{ni}(x)$:

(H_1) $\sum_{i=1}^{n}\mathsf{E}\left(\omega_{ni}(x)\right)\to 1, n\to\infty$;

(H_2) $\sum_{i=1}^{n}\mathsf{E}\left(|\omega_{ni}(x)|\right)\leqslant C<\infty$, for all n;

(H_3) $\sum_{i=1}^{n}\mathsf{E}\left(|\omega_{ni}(x)|\right)\cdot|h(x_{ni})-h(x)|\,I(\|x_{ni}-x\|>\alpha)\to 0, n\to\infty$, for all $\alpha>0$.

We can see that the conditions (H_1)–(H_3) are general assumptions. For example, Priestley-Chao weight

$$\omega_{ni}(x)=\frac{x_i-x_{i-1}}{r_n}K\left(\frac{x-x_i}{r_n}\right)$$

and Gasser-Müller weight

$$\omega_{ni}(x)=K\left(\frac{x-x_{ni}}{r_n}\right)\Bigg/\sum_{j=1}^{n}K\left(\frac{x-x_{nj}}{r_n}\right),$$

where $K(\cdot)$ is a bounded density function, r_n is the bandwidth. Based on the assumptions above and Corollary 15, the complete consistency of the nonparametric regression estimator $h_n(x)$ is obtained as follows.

Theorem 16 Let $1 \leqslant l < 2$, $\{\epsilon_n, n \geqslant 1\}$ be a sequence of identifically distribution NSD random errors with $\mathsf{E}(\epsilon_1) = 0$, which is stochastically dominated by a random variable X satisfying $\mathsf{E}\left(|X|^{2l}\right) < \infty$. Suppose that $\{\omega_{ni}, n \geqslant 1, 1 \leqslant i \leqslant n\}$ is rowwise independent and independent of $\{\epsilon_i, 1 \leqslant i \leqslant n\}$. If conditions (H_1)–(H_3) hold and satisfy

$$\sum_{i=1}^{n} \mathsf{E}\left(|\omega_{ni}(x)|^q\right) = O(n^{1-q/l}),\ \text{for some } q > 2l/(2-l), \tag{12}$$

then for all $x \in c(h)$,

$$h_n(x) \to h(x),\ \text{completely.} \tag{13}$$

Proof For any $\alpha > 0$ and $x \in c(h)$, by (10) and (11) it follows that

$$\begin{aligned}|\mathsf{E}(h_n(x)) - h(x)| \leqslant & \sum_{i=1}^{n} \mathsf{E}(|\omega_{ni}(x)|) \cdot |h(x_{ni}) - h(x)|\, I(\|x_{ni} - x\| \leqslant \alpha) \\ & + \sum_{i=1}^{n} \mathsf{E}(|\omega_{ni}(x)|) \cdot |h(x_{ni}) - h(x)|\, I(\|x_{ni} - x\| > \alpha) \\ & + |h(x)| \cdot \left|\sum_{i=1}^{n} \mathsf{E}(\omega_{ni}(x)) - 1\right|.\end{aligned} \tag{14}$$

We obtain from $x \in c(h)$ that for all $\epsilon > 0$, there exists a constant $\delta > 0$ such that for all x' statisfying $\|x' - x\| < \delta$, we have $|h(x') - h(x)| < \epsilon$, and thus by assumptions $(H_1) - (H_3)$ and the arbitrariness of ϵ we have

$$\lim_{n\to\infty} \mathsf{E}(h_n(x)) = h(x). \tag{15}$$

In order to prove (13), it suffieces to show for all $\epsilon > 0$ by (15)

$$\sum_{i=1}^{\infty} \mathsf{P}\left(\left|\sum_{i=1}^{n} \omega_{ni}(x)\epsilon_i\right| > \epsilon\right) < \infty. \tag{16}$$

Applying Corollary 15 with $X_i = \epsilon_i, a_i = \omega_{ni}(x)n^{1/l}, s = 1$, (16) holds immediately. The proof is completed. $\square$

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负超可加相依随机变量 Sung 型加权和的完全 f 矩收敛性

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摘 要: 利用负超可加相依随机阵列（NSD）的 Marcinkiewicz-Zygmund 型不等式和 Rosenthal 型不等式及截尾的方法，探讨了 NSD 随机变量 Sung 型随机加权和的完全 f 矩收敛性，在一些合适的条件下，获得了 NSD 随机变量 Sung 型随机加权和完全 f 矩收敛性的一些一般结果，推广并改进了相关文献已有结论。最后，给出了基于 NSD 误差的非参数回归模型中随机加权和估计的完全相合性例子作为其应用。

关键词: Marcinkiewicz-Zygmund 型不等式; Rosenthal 型不等式; Sung 型随机加权和; 负超可加相依随机变量; 完全 f 矩收敛

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